Teaching and learning across the A level - university transition

Hannah Lewis
Department of Mathematical Sciences
University of Bath

ma9hl@bath.ac.uk

Jane White
Department of Mathematical Sciences
University of Bath

kajw@maths.bath.ac.uk

As part of a growing collaboration between local schools/colleges and the Department of Mathematical Sciences, University of Bath, a small scale study was carried out to explore differences and similarities in teaching probability and calculus in the two institutions. More generally the collaboration was initiated to develop and enhance methods to smooth the transition from A level to undergraduate studies in the mathematical sciences.

There is an increasing interest in developing ways to tackle the mismatch in the mathematical skills which new undergraduates possess and what is required of them by their courses. This problem is not restricted to mathematics degrees alone (although this study is), but has some effect on many courses with a mathematical content.

Diagnostic testing followed by remedial assistance is one common method employed by many university courses to help to ease the problem. In general, however, little attention has been paid to differences in teaching and learning styles across the transition although these must surely influence student success in making the transition. The work in Bath has tried to focus on these aspects of the transition, believing them to be very important. One example of this work is a collaboration between school teachers and lecturers, which has resulted in a booklet about teaching proof across the transition, looking in some depth at notation and general formalism of mathematics in the two institutions.

Part I : Introductory probability

The investigation into teaching and learning in probability actually combines the idea of teaching content and style. Nowadays, most mathematics undergraduates have studied one or more statistics modules at A level. Because there is no requirement in the core A level for students to study these modules, many universities decide to teach probability from scratch. Given the increase in formalism that is used at undergraduate level, this is often an appropriate use of time but it does mean that some thought must be given to what students have already encountered so that their experiences can be built on, rather than existing knowledge confusing the situation.

As well as the obvious increases in formalism, it was found that many of the problems presented to undergraduates required different styles of solution to those presented at A level. Typically what was required was either an increase in the rigour of the argument, more notation or greater interpretation of the results. This highlights the difficulty of presenting familiar material to undergraduates - they already have a successful framework for dealing with the material to answer questions and unless the new information or formalism complements this, students often find it hard to understand why they can no longer answer questions that they could at A level.

To illustrate the differences that are common across the transition, we present some extracts taken from first year undergraduate studies in probability and from A level statistics and probability modules. Obviously there could be significant variation in how this material is presented depending on the style of the teacher or lecturer. Our comments, however, are consistent with many other observations which have been made as part of this study.

Extracts 1A and 1B both cover how to calculate the probability of an event when all events are equally likely, and even though they are short, comparison shows many differences in the areas discussed. Extract 2 presents solutions to a simple probability question as seen at university and at school level and chosen from a selection of undergraduate questions.

Extract 1A: lecture notes from an undergraduate unit.

Theorem

Let $\mathbf{\Omega}$ be the sample space of an experiment with $|\mathbf{\Omega}| < \infty$ and where each outcome is equally likely, then for any event $A \subseteq \mathbf{\Omega}$

$$P(A) = \frac{|A|}{|\Omega|}$$

where for any event B, |B| is the number of outcomes in event B.

Examples

Toss a fair coin three times. What is the probability we get at least two heads?

Sample space:

$\mathbf{\Omega}$ = {HHH, HTH, HHT, HTT, THH, TTH, THT, TTT}

So $|\mathbf{\Omega}| = 2^3 = 8$ possible outcomes.

Let A be event get at least two heads, so

A = {HHH, HTH, HHT, THH} and |A| = 4

Then as outcomes are equally likely (the coin is fair)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$$

Extract 1B: An abstract from an A level lesson. This is from the initial lesson covering probability.

Probability

ξ = everything

the probability of an event E is defined

$$P(E) = \frac{\textit{number of ways E can happen}}{\textit{total number of possibilities}} \left(= \frac{n(E)}{n(\xi)} \right)$$

provided all possibilities are equally likely.

Examples

- A dice is thrown. The probability of a prime number is

  P (prime) = $^3/_6$ = $^1/_2$
- Two coins are tossed. What is the chance of 2 heads?
  Possibilities are: *HH HT TH TT*

  P(*HH*) = $^1/_4$

At first sight the way in which the probability definition is presented in these two extracts is very similar. However, there are potentially significant differences which pervade throughout the definitions. For example, in the university definition, the sample space is restricted to be finite, the notation $|A|$ is used to describe the number of outcomes in event A, where A is defined to be a subset of the sample space. In comparison, less notation is used within the school definition, there is no explicit restriction on the size of the sample space, and there is no statement to ensure E is a subset of the sample space. The consistent use of notation is clearly demonstrated in the example which follows the university definition with explicit use of set notation to describe the sample space and the required event. The example used to demonstrate the probability of an equally likely event in the school extract does not directly relate back to the definition - rather it uses the definition intuitively or does partial listings, such as in the second example where the event possibilities are listed but not the sample space.

Different levels of rigour are also apparent if the extracts are viewed in their original context: the way in which the lecturer arrived at the theorem (stating axioms of probability and proving consequences which lead to the theorem) was much more rigorous than the intuitive reasoning used to arrive at the same formula in the A level material. The differences in levels of formalism are also apparent in the problem solutions of Extract 2:

Extract 2

Question

A family has two children. Given that at least one of the children is a boy, what's the probability that both are boys?

A level solution

Two children families, assume P(B) = P(G) = ½ and gender of the children is independent.

Families are BB, BG, GB, GG

$$\mathrm{P(BB \mid at\ least\ 1\ boy)} = \frac{\mathrm{P(BB} \cap \mathrm{at\ least\ 1\ boy)}}{\mathrm{P(at\ least\ one\ boy)}} = \frac{\mathrm{P(BB)}}{\mathrm{P(at\ least\ one\ boy)}}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Undergraduate level solution

Let Ω = {(BG), (BB), (GB), (GG)} where (BG) means oldest is a boy, youngest a girl etc.

Assume all outcomes are equally likely,

Let E be event both are boys, so E = {(BB)}

Let F be event there is at least one boy so F = {(BG), (BB), (GB)}

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{|E \cap F|}{|F|} = \frac{1}{3}$$

The solution from the A level perspective this time lists the sample space but not the event possibilities. The notation and structure of the university solution is consistent with the theory presented and demonstrates how the general idea of probabilities for equally likely events can be translated into a more complex example. In some senses, there seems to be an assumption that students can make the connection between theory and example and can pick out the particular event set that they need. Interestingly some of the proposed questions were dismissed for inclusion in the study because it was felt that they were not accessible to the majority of A level students for some reason; most commonly the rejected questions required some assumption to be made which was not entirely obvious.

A final point worth mentioning about both sets of extracts included here, is that the notation which is used at undergraduate level does vary significantly from that used in the school extracts. Provided that new notation is introduced carefully and in respect of students existing experiences, this should prove no real problem. However, it can be difficult to gauge exactly what all students have been used to at school level and so a consistent approach which is thoughtfully introduced is probably the most appropriate way to ensure students remain confident with material that they have already seen.

Part II : Introductory Calculus

The main problem faced when attempting to compare the way in which calculus is taught at the two levels is that there is very little common material. Unlike with probability, where some sections are often re-taught at undergraduate level, the area of calculus builds on work covered at A level.

There are some examples of shared material, such as differentiation which is covered in some detail in A level courses and is revisited in two first year units at the University of Bath. One of these considers differentiation from first principles, the other reviews differentiation covered at A level (such as product, quotient and chain rule). It is material that is covered in this unit which has generally been used for comparison with A level material.

The pattern which emerges from observing the lessons and lectures seems to suggest that schools and universities have very different approaches to teaching introductory calculus. In general, A level lessons approach new material by considering a specific example and then generalising; undergraduate courses on the other hand begin by giving the general case and then qualifying this with an example.

Introductory lessons on differentiation were observed at three different schools as part of this study. All three used the same approach: considering the gradient of a straight line and then evaluating the gradient of the curve $y=x^2$ at specific points.

This is demonstrated in Extract 3, where the teacher considers two general points P and Q on the curve $y=x^2$. Although the points are general it is an interesting point to note that the curve is not.

Extract 3: from school notes

$$\text{Gradient chord } PQ=\frac{\delta y}{\delta x} \quad y=x^2 \quad P(x,x^2), \quad Q\left(x+\delta x,\left(x+\delta x\right)^2\right)$$

$$\begin{aligned}\text{Gradient of chord PQ} &= \frac{\left(x+\delta x\right)^2-x^2}{x+\delta x-x}\\ &= \frac{x^2+2x\delta x+\left(\delta x\right)^2-x^2}{\delta x}\\ &= 2x+\delta x\end{aligned}$$

$$\text{Gradient at P} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=2x$$

Specific cases were used to illustrate the point, or more specifically to help students to `spot a pattern', and then generalise. In contrast, the general function f(x) was used in the two undergraduate courses as shown in Extract 4.

Extract 4: from undergraduate notes

$$f'(x)=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]=\lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{h\delta x}\right]$$

Of course this could be because the lecturers know that the students have already been introduced to differentiation. However in the lessons and lectures observed the trends seem to continue.

Integration was introduced in a similar way. In one school, students' notes state that:

Extract 5: from school notes

We may consider integration as the reverse process of differentiation, ie given the gradient function, what is the initial function? ...

e.g. if we know that $\frac{dy}{dx} = 3x^2$, then $y = x^3 + C$

This is called the indefinite integral.

A new type of notation is often used.

We write $\int f' dx = f(x) + C$

This contrasts with the general and more formal university approach seen in Extract 6:

Extract 6: from undergraduate notes

The process of finding anti-derivatives is integration so if

$$\frac{d}{dx}[F(x)] = F' = f(x)$$

then $\int f(x)dx = F(x) + C$ ...

There are many other examples which conform to these `typical' approaches, such as integration by substitution and differentiation of a sum. It can be argued that that this is done because students are being introduced to the material at A level and are already familiar with it by the time they are undergraduates. If this were the case one would think that when the same lecturers were covering material new to students they would adopt the first approach. This is not the case. Throughout the undergraduate material (pure mathematics) the most common approach to any new material is to give first a general theorem (and proof) followed by an example: in short the approach is general $\rightarrow$ specific, which contrasts a common school approach, namely specific $\rightarrow$ general.

This is a very brief overview of what we have been engaged with at the University of Bath. This study was small scale and does not pretend to have found a perfect solution to problems faced at the transition when teaching statistics. However, one of the objectives of the study was to raise awareness of exactly how different and similar teaching and learning is between school and university and to explore whether the same material is presented in significantly different ways. This has been achieved to some extent and is discussed in the full report of this study which is available from the Department of Mathematical Sciences on request*. In conclusion, universities should acknowledge what has already been learnt by students in schools, and to work with this background to enhance the learning which takes place in Higher Education.

* Editor's note: Any correspondence relating to this article should be directed to Jane White, email kajw@maths.bath.ac.uk