Book Reviews

Title: An interactive introduction to Mathematical Analysis
Author: Jonathan Lewin,
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It was the word 'interactive' which made me curious about this presentation of real analysis. The 'interaction' referred to is with a CD which accompanies the book and indeed it is the CD which makes this book distinctive. The CD contains an electronic version of the book, and much more. For chapters 4 (set theory), 6 (the topology of the real line), 7 (sequences), 8 (limits of functions and continuity) and 11 (the Riemann integral) the CD contains not only the book version of each chapter, but also a more advanced, topological/metric space, version as well. Chapters 15 (complex variable), 17 (sets of measure zero) and 18 (calculus of several variables) only exist on the CD. The book itself contains no worked solutions or hints to the exercises. The CD contains hints and solutions to all the exercises (some of which are only available at the discretion of the 'instructor') and supplementary exercises. The CD also contains solutions to material declared in the book to be 'easy' or 'left as an exercise for the student'. Most importantly the CD contains *Scientific Notebook*, a word processing package combined with the computer algebra packages MuPAD and MAPLE. The full *Scientific Notebook* is available after installation, for 30 days. Thereafter it degrades to the freely downloadable *Scientific Viewer*, or, with registration and a fee, may be installed permanently. There are frequent indications beside the exercises in the book either that an exercise should be tackled with the help of *Scientific Notebook*, or that the CD contains hints or an open solution.

Finally the CD contains 'mini-lectures' covering some 50 sections of the book. In these, the screen reveals the text of the book, line by line. The author, with what I imagine to be a South African accent, talks through the arguments in great detail using the cursor to point. The user can accelerate through or repeat parts of the mini-lectures at will. It should be said that apart from the lack of solutions the book makes sense by itself, and the computer algebra facility has not been used to reshape the real analysis syllabus, which is in fact conventional.

The author's style in his mini-lectures - very detailed and precise - is evidenced in many places in the main text, and theorems are helpfully followed by a list of significant examples and insightful comments. Both these aspects are good for beginners. But rather strangely, definitions tend to appear in abstract form and almost without motivation: for example, $x_n \to x$ as $n \to \infty$ is defined to mean, for any neighbourhood U of x, x_n is eventually in U. Also the rate of increase of difficulty in exercises is pretty steep. Significant easier exercises to feed the intuition are missing. With the exercises, the author covers his tracks with links to assistance on the CD. But perhaps the real explanation is that this book has been written for an American market, where all students opting for a real analysis course will have already completed a calculus course in which ε-δ proofs play a significant role. This book contains all the definitions, but not all the preliminary exercises which a student beginning rigorous work with limits will generally need.

To run through the chapters, chapter 1 is historical, on which I will comment later. Chapter 2 is on quantifiers, negation and contrapositive. The basis for the chapter is mostly common sense, but the questions which relate well to the book as a whole are pretty tough. My own hunch is that such material would be better integrated with the course, rather than separated out as a preliminary. I am not sure that clarity about who wears top hats in a room, improves anybody's real analysis. I suspect that the same remarks apply to chapter 3 on implication and contradiction and chapter 4 on set theory and the definition of function.

With chapter 5 on the real number system, the course properly begins. There is an unusual song and dance about the distinction between algebraic numbers and surds (surds have to have real numbers under their roots) and then we are into the axioms. The order relation is given a modest treatment and then there is a lengthy lead in to the notion of least upper bound. Strangely, there is no motivation for the least upper bound axiom except the remark "almost every major theorem of calculus depends strongly on a 'completeness property' of the system **R**". The discussion of upper bounds includes no mention of the distinction between **Q** and **R**. Occasionally, in the text that follows, the dependence of some theorem on completeness is noted. In no case is independence of completeness noted. Secrecy in this matter does not bring clarity. In chapter 9, two theorems about the differentiation of inverse functions, which look very similar, are placed side by side. Theorem 9.3.8 does not need completeness, while 9.3.9 does. The distinction is overlooked. Back in chapter 5 Lewin claims that "the Archimedean property of **R** depends on the axiom of completeness. Without this axiom, the Archimedean property may actually be false." The author might have added: "without this axiom the Archimedean property may actually be true", if only he had considered the rational numbers! Chapter 5 ends by adjoining ∞ and $-\infty$ to **R**. Induction is given an optional section on the CD.

Chapter 6 is about open and closed sets in **R**, limit points and neighbourhoods of infinity. Chapter 7 is about limits of sequences. Because **R** has been extended, all monotonic sequences have a limit. Cauchy sequences, lim sup and lim inf are optional matters to be found only on the CD. Some of the more complicated sequences which appear in the exercises are defined iteratively, and their treatment would have been rounded off by a discussion of fixed point iteration, which, sadly, I could not find in the book. Chapter 8 begins with a conventional treatment of limits of functions. Only in the later part of this chapter, the part on continuity, is there a link made with the sequences in chapter 7. Here the intermediate value theorem is proved and uniform continuity is introduced. Chapter 9 goes from the definition of derivative up to Taylor Series. Chapter 10 gives a formal development of exponential and logarithmic functions. This includes finding the limit of $\sqrt[n]{a}$ as $n \to \infty$ and the limit of $(a^h - 1)/h$ as $h \to 0$, but not (oddly enough) the limit of $(1 + 1/n)^n$ as $n \to \infty$. This is strange since $(1 + 1/n)^n$ is investigated in 7.7.4 qn 5, one is within an ace of the limit with 10.6.6 qn 7 and it is needed for 10.6.6 qn 9.

Chapter 11 on the Riemann integral starts with step functions. I doubt there is a subject more in need of motivation than step functions, and this could have been done with Fermat's integral of the quadratic function which could have been used to introduce the subject of integration but is held off till 11.6.2. The chapter includes an unexpected dip into measure theory. With chapter 12 on Infinite Series, one gets the impression that this is where the author is happier. The motivation is better here. There are many difficult exercises using d'Alembert's and Raabe's tests. 12.6.2 qn 2 seems to be wrong and 12.6.12 qn 15 (b) asks for the (false) converse of a true result (the effectiveness of d'Alembert's ratio test implies the effectiveness of Cauchy root test for convergence, but not conversely). Chapter 13 is on improper integrals and chapter 14 on uniform convergence.

It must be said that while the author's explanations are rigorous and careful, his motivations are thin and his preference for slick logic rather than food for the intuition really suit a *second* course. For example, he only discusses the convergence of $\Sigma 1/n$ and $\Sigma 1/n^2$ with the help of the Integral test.

Lastly, some remarks on history. It is good that the author has tried to give a historical overview of real analysis in chapter 1, and I welcome his historical quotations. I have no doubt that history provides the key to unpacking the, now conventional, logical sequence of presentation, in favour of something which more readily engages a student's intuition. But here Lewin misses some great opportunities with his use of history. There are historically dubious elements in his description of the 'Pythagorean crisis'. But the significance of the episode is that every mathematician has an analogous experience in first grappling with irrationals. Later this affects the student's understanding of convergence, limits, continuity and differentiability. Apart from three unexplained "facts" listed in section 9.1.1 the 'crisis' is ignored throughout the book. The paradoxes of Zeno (section 1.3) expose the distinction between limits and equality. But the fact that the concept of limit resolves the paradoxes is not mentioned in either chapter 7 or 8 despite the fact that the problems which today's students have with limits include the issues highlighted by Zeno. In no sense do Zeno's paradoxes highlight the completeness of the real numbers (which one reading of page 9 might be thought to suggest). The difficulties of the paradoxes are all real enough once rational numbers are allowed. Lewin quotes from Bishop Berkeley, from Voltaire, from Cantor and from Frege. Would that he had woven his story from the writings of Newton, Euler, Gauss, Cauchy and Dedekind. The historical steps are ones that humankind took using guesswork and intuition. The mathematical growth of students today is likewise a human event. The logical sequence can only be devised by those who have already intuitively grasped the mathematical development.

The interrelation of book and CD which Lewin offers is full of potential. The interplay of computation with pencil and paper algebra is one that we teachers are all struggling to take advantage of. I see Lewin's book as a springboard for a successor rather than a book to recommend to British universities today.