# InTeχ: a problem-centred approach to learning integration techniques

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Students are reasonably competent at applying the various integration techniques, but are frequently unable to choose an appropriate technique for a given integrand. To overcome this difficulty, a problem-centred approach has been devised, whereby integrands are classified into categories which are then identified as requiring particular techniques. This has been implemented on the Web, using hyperlinks for the classification process, leading to detailed descriptions of the techniques and worked examples of their use. This resource, known as InTeχ, has been trialled with a class of 156 first-year mathematics students at Loughborough University, the majority of whom found it useful; however, it is not yet clear whether InTeχ has actually improved their skill in choosing integration techniques or whether they have used it as a crutch.

***The difficulty***

Differentiation is an algorithmic process: given the derivatives of a few basic functions, together with a few rules (the addition, product, quotient and chain rules are sufficient), it is possible in principle to differentiate any function, however complicated, formed by combining the basic functions in any way. In contrast, integration (or, more precisely, antidifferentiation, the usual analytical procedure by which we evaluate integrals) is not algorithmic: there is a toolbox of techniques (substitution, parts, etc.), but it requires experience to know which technique will enable one to antidifferentiate any particular integrand. Over 5 years of teaching a one-variable calculus module to first-year mathematics (single and joint honours) students at Loughborough University, it has been my observation that students are reasonably competent at using the various techniques; however, unless given some direction as to which technique to use, they frequently fail to evaluate all but the simplest integrals because they start off by choosing the wrong technique.

Why is it so difficult to choose the correct technique to antidifferentiate a given integrand? The following examples will illustrate the difficulty.

***Example 1***

Consider the three integrals:

(a) $\int \frac{1}{x^2+4x+3}\,dx$

(b) $\int \frac{1}{x^2+4x+4}\,dx$

(c) $\int \frac{1}{x^2+4x+5}\,dx$

The integrands differ only in the value of a constant, and might reasonably be expected to require the same technique to antidifferentiate them. However: the denominator in (a) can be factorised, so the integrand can be re-written as partial fractions in order to antidifferentiate it; the denominator in (b) can be recognised as an exact square, and the antidifferentiation is then immediate (or may require the substitution $u = x + 2$ for a very inexperienced student); the denominator in (c) is an irreducible quadratic, and completing the square yields a form which can be recognised as the derivative of an arctan function

*Visit the Inteχ web pages at:*

**http://learn.lboro.ac.uk/ma/01maa140/content/InTech/**

(or may require the explicit substitution $x+2 = \tan\theta$ for an inexperienced student).

Three different techniques for three very similar integrals; and it is no good pointing out that completing the square in (a) yields a form recognisable as the derivative of an arctanh function (cf. arctan in (c)), or that (c) can be evaluated by finding the complex factors of the quadratic and using partial fractions: no reasonable person would use these techniques, at least when teaching undergraduates.

***Example 2***

Consider the integrals,

(a) $\int \exp(-x^2)dx$ (b) $\int x\exp(-x^2)dx$

Here (a) is a classic example of an integral which looks very simple but cannot be evaluated at all in terms of elementary functions. Integral (b) looks more complicated than (a), but is in fact simple to evaluate using the substitution $u = -x^2$. However, consider the thought processes of many students when confronted with (b):

*$x\exp(-x^2)$ is a product, so I need to integrate by parts. I can integrate x, but I can't integrate $\exp(-x^2)$, so that fixes my choice of "u" and "v" in the integration-by-parts formula. Since*

$$\int x dx = \frac{x^2}{2} \quad \textit{and} \quad \frac{d}{dx}\exp(-x^2) = -2x\exp(-x^2)$$

*I get*

$$\int x\exp(-x^2)dx = \frac{x^2}{2}\exp(-x^2) - \int \frac{x^2}{2}.(-2x\exp(-x^2))dx$$

*Oh dear, the new integral on the right-hand side is worse than the one I started with! I give up.*

This example illustrates how some of the ideas about integration that are commonly taught to students (in this case, the idea that "products should be integrated by parts") are in fact of limited validity. It reinforces the message that the only way to become fluent in techniques of antidifferentiation is by practice: students need to work on large numbers of examples. But how will they get that practice if they become demoralised by repeated failure as in the example of $\int x\exp(-x^2)dx$ given above?

***An idea to overcome the difficulty***

Textbooks and lecture courses generally expound the various integration techniques sequentially, giving examples of the use of each technique; it is difficult to conceive of any other procedure for teaching the subject in print or in lectures. However, existing CAL packages and web-based calculus tutorials also tend to adopt this one by-one approach to integration techniques, even if it is here possible to choose any technique to study at any time. The essence of these learning resources is that they are "technique-centred"; but what a student doing coursework (or indeed a worker using mathematics in a "real-life" situation) needs when confronted with an integral is a "problem-centred" resource. Such a resource would demonstrate how to antidifferentiate various types of integrand, rather than how to use various techniques.

The Web with its hyperlinks is an ideal medium for creating such a resource. The basic structure is a "decision tree" by which an integrand can be classified, with the classification process continuing until the integrand is categorised sufficiently tightly that an integration technique can be assigned. For instance, the integrands in Example 1 are first classified as "algebraic" (as opposed to "trigonometric", etc); within the algebraic category, they are then classified as "rational functions" (as opposed to polynomials, functions involving fractional powers, etc.); within the category of rational functions, they are classified as having quadratic denominators; finally, they are classified according to whether the quadratic can be factorised, and at this stage the appropriate technique (partial fractions, or completing the square and using a tan substitution) is explained. This procedure works through web pages with hyperlinked lists of categories of integrands, leading to further pages with lists of sub-categories, and finally to pages explaining in detail how to antidifferentiate each type of integrand.

This resource has been created under the name InTeχ, and contains some introductory pages explaining its purpose and how to use it, leading to a "List of types of integrand" which is the base of the decision tree. Here, and on each page higher up the tree on which users need to choose from a list of categories of integrand, the link for each category is accompanied by a link to "Examples of this category" for any user who may be unsure of what a category title means. For instance, by following the "Examples of this category" link below "Rational functions" on the page listing categories of algebraic functions, one will arrive at a page which first explains that a rational function is a quotient of polynomials, and then lists some typical rational functions, arranged roughly in increasing order of complexity. Many of

these pages also include some "Examples of integrands not in this category" to further reduce the scope for misunderstanding. The pages which explain integration techniques in detail all carry links to a page of "Worked examples of integrating ... ", on which the examples are again arranged in roughly increasing order of complexity, starting with a very simple one and going on to an example at least as complicated as any undergraduate is likely to be faced with. Indeed, the site is far more comprehensive than our students are likely to need (how often will they be asked to integrate a product of powers of sech and tanh functions?); just about anything that is composed of elementary functions and has an antiderivative in terms of elementary functions is covered. InTeχ may be found at
http://learn.lboro.ac.uk/ma/01maa140/content/InTech/

***Bringing the idea to fruition: the lecturer's experience***

The opportunity to create this resource came when Loughborough University obtained funding from the HEFCE's Teaching Quality Enhancement Fund for a secondment project, whereby staff time could be bought out in order for them to develop or extend innovative teaching in their modules. Training in production of web-based resources was provided for staff who were inexperienced in this (like myself). As it turned out, much of my secondment period (two days per week for a 15-week semester starting in September 2000) was occupied with overcoming technical problems. The majority of pages were written using IBM techexplorer, a relatively new system for writing in LATEX code for Web publishing, which produces pages whose appearance is very dependent on which platform is used, and indeed on exactly how one's computer and monitor are set up.

Having overcome these technical difficulties (and listed those which I was unable to overcome on a page linked from the InTeχ Welcome page), I proceeded to complete the 200+ pages that constitute InTeχ in time for the Calculus 1 module (for single and joint honours mathematicians) in semester 1 of the 2001/2 academic year. InTeχ was demonstrated to students during lectures on integration techniques; but to ensure that students would actually use it, it was necessary to make their assessment depend on it; simply encouraging them to use it on an example sheet was not enough. Thus, one of their coursework assignments was set requiring them to evaluate two integrals of types not covered by any of the examples done in lectures. An email questionnaire was prepared (see Appendix), and was sent to all students on the module a few days after the handing-in date for the coursework.

***The student experience of InTeχ***

Unfortunately, the first experience for many students was of technical problems. IBM techexplorer is a plug-in, with a free Introductory Edition which had been installed and tested in all undergraduate computer laboratories at Loughborough. A link to the IBM techexplorer web site was also provided from the InTeχ Welcome page for students who wished to install it on their own computer. On the day after the coursework requiring InTeχ was set, I received my first email from a student complaining that she had been unable to view pages written using IBM techexplorer on her own computer. This did not surprise me too much, since it had taken me two attempts to install it correctly on my own machine. However, much more worrying was a report from another student a few hours later that she had been unable to view IBM techexplorer pages on a computer in a university laboratory. More similar reports soon followed, while other students reported having no problems at all, either in university laboratories or on their own computers. After much desperate communication with the university's Computing Services staff, it eventually transpired that the apparently random discrimination between students as to whether IBM techexplorer would work for them was related to whether they were using a secure access to the server on which InTeχ is installed (but it has never been explained to me why secure access should prevent IBM techexplorer from working!). This was only ascertained one day before the handing-in date for the coursework, so tutors were asked to make allowances when marking it. The lesson from all of this is that all students should have been asked to test whether they could read InTeχ at least a week before they needed to use it in earnest for coursework; however much testing the teaching staff and Computing Services staff do, you can never guarantee that the students won't find problems!

Despite all these difficulties, the students in general did the coursework assignment well. Nevertheless, I was rather dreading the responses when I emailed the questionnaire a few days later. 26% of the students taking the module (40 out of 156) responded, and the results were more encouraging than I could have hoped for. 31 out of the 40 respondents had used InTeχ, with 8 of the 9 non-users saying that this was due to the problems with viewing IBM techexplorer pages (the remaining one saying that he had been able to do the examples and coursework without InTeχ); only 11 respondents said that they had no computer-related difficulties at all. The demonstrations of InTeχ in lectures were found useful, but some respondents found that it became tedious seeing InTeχ used for every example in four lectures; they would have preferred to have a hands-on demonstration in a computer laboratory (but Loughborough hasn't got a lab big enough to take 156 students at the same time!).

Of those who had made some use of InTeχ, only 4 respondents had not used it for the coursework assignment, while 11 had not used it to help with the (non-assessed) problems on the Integration Techniques example sheet; some of the non-users said that they found it easier to use textbooks. With regard to ease of finding one's way to the information needed, 25 out of 30 respondents to this question found it very easy or quite easy. The difficulties that students noted were generally related to the comprehensiveness of the resource, which inevitably means that there are large numbers of options to choose from, sometimes leading to confusion; furthermore, relatively simple cases sometimes require an inordinate amount of work to arrive at the explanation of how to perform the integration. However, once these explanations had been arrived at, 24 out of 32 respondents found them to be very clear, though there were again some comments that my insistence on comprehensiveness had led to some of the integration techniques appearing more difficult than they actually are in practice.

Possibly the most gratifying result was in response to a question as to whether students were likely to use InTeχ to help with integration in the future, with 17 replying "Very likely" and only 3 saying "Not very likely" or "Not at all likely". There is an unfortunate tendency for many students to compartmentalise mathematics, not realising that the different topics they learn are related both horizontally and vertically, i.e. that there are connections between different topics at the same level, and that what they learn at early stages in their studies will be used in higher level topics. While the wording of my question specifically points out the vertical connection, the students' positive response at least shows some understanding of this issue.

### *Reflections on the value of InTeχ*

I make no claim that InTeχ is a panacea for students' difficulties with integration. In fact, there is a danger that it may become a crutch, leaving students helpless when it is taken away: one student commented that its drawback was that it would not be available in the exam room. It certainly does not reduce the amount of practice students need in order to become fluent in integration techniques; on the contrary, it should encourage them to practice more, by reducing the scope for becoming demoralised through spending ages going down blind alleys. Although the coursework was designed so that students would need to use InTeχ (or else spend considerably more time looking through a textbook to find the appropriate techniques), the non-assessed example sheet encouraged them to first try the examples without InTeχ and to use InTeχ only if they got stuck. While students have certainly found it useful in helping them to evaluate integrals, it is not at all clear how well they have learnt how to choose and apply integrations techniques; maybe the large number saying that they were likely to use InTeχ in future indicates that they still need the crutch.

With the well-documented decline in university entrants' mathematical knowledge and skills, there is also an issue as to whether InTeχ is appropriate to a first-year, first-semester module. Although basic integrals are included, InTeχ is mainly designed to help with evaluating more complicated cases, of the sort which it is becoming increasingly inappropriate to ask students at this level to do. More fundamentally, one may ask whether in the age of computer algebra we should expect students to learn techniques for evaluating such integrals at all. At Loughborough we teach our first year students how to use Maple, so should we expect them to be able to do any kind of manipulation that requires more time to do on paper than by means of this computer algebra system? Are the technical skills of differentiation and integration as outdated as arithmetical skills such as long multiplication (for which we all use calculators)?

If so, what is there left to teach in Calculus? The concepts of differentiation and integration, limits, stationary points, Taylor series, etc. are certainly still vital, but practical skills of evaluation may no longer be important. Yet this will leave us teaching pure analysis, which university entrants are even less well prepared for than the current Calculus syllabus, and I cannot visualise any InTeχ-style resource that will help with that problem.

### *Appendix*

The questionnaire reproduced below was emailed to students on Monday 10th December 2001, following the coursework assignment on integration techniques which they had to submit on the previous Thursday. Where questions ask students to choose from a list of possible responses, the numbers of students giving each response are shown in brackets. A total of 40 students returned the questionnaire, but not all respondents answered all questions.

***To all students on module MAA140 Calculus 1:***

InTeχ, the Integration Techniques Web site, has been introduced this year so you are the first group of students to use it. I would therefore like some feedback from you so that I can improve it for subsequent years. Could you please spare a few minutes to complete the questionnaire below? Just use the Reply facility on your email. Then, where a question gives you a set of options (e.g. Yes / No or Very useful / Quite Useful / Not very useful / Totally useless), simply delete all options that don't apply, leaving one option. Where a question asks you for comments, simply write your comments below the question.

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1. (a) Have you used InTeχ at all?
Yes [31] / No [9]

(b) If "No", was this because of technical problems with accessing InTeχ or for some other reason? Please write the reason below.

2. (a) How useful were the demonstrations of InTeχ during lectures?
Very useful [13] / Quite useful [20] / Not very useful [4] / Totally useless [0]

(b) If you didn't reply "Very useful", please say what were the bad points of the demonstrations of InTeχ during lectures, and give any suggestions for improvement. If you answered "No" to 1.(a) above, go straight to question 8 now.

3. (a) When using InTeχ, was it easy to find your way to the information you needed?
Very easy [11] / Quite easy [14] / Not very easy [5] / Very difficult [0]

(b) If you didn't reply "Very easy", please say what were the bad points of the way InTeχ is organised, and give any suggestions for improvement.

4. (a) When you arrived at pages giving the information that you needed, was the information presented clearly?
Very clearly [24] / Quite clearly [7] / Not very clearly [1] / Not at all clearly [0]

(b) If you didn't reply "Very clearly", please say what were the bad points of the presentation, and give any suggestions for improvement.

5. (a) Did you use InTeχ to help you answer questions on examples sheets (especially Examples 6)?
Yes [21] / No [11]

(b) If "No", why not?

(c) Did you find InTeχ useful in helping you to answer questions on examples sheets?
Very useful [11] / Quite useful [6] / Not very useful [0] / Totally useless [0]

(d) If you didn't reply "Very useful", please say why InTeχ wasn't a great help, and give any suggestions for improvement.

6. (a) Did you use InTeχ to help you do Coursework Assignment 5?
Yes [23] / No [4]

(b) If "No", why not?

(c) Did you find InTeχ useful in helping you to do Coursework Assignment 5?
Very useful [12] / Quite Useful [7] / Not very useful [3] / Totally useless [3]

(d) If you didn't reply "Very useful", please say why InTeχ wasn't a great help, and give any suggestions for improvement.

7. You will need to do integration in various Mathematics modules throughout your programme at Loughborough. Are you likely to return to InTeχ to help you with integration in the future?
Very likely [17] / Quite likely [9] / Not very likely [2] / Not at all likely [1]

8. (a) Did you have any organisational or technical problems in using InTeχ (e.g. unable to find a computer available when you needed it, problems with viewing InTeχ pages)?
Yes [25] / No [11]

(b) If "Yes", please specify the problems you had.

Please write any further comments on InTeχ below. Then send your completed questionnaire back to me (a.kay@lboro.ac.uk) using the Reply facility on your emailer.

Thank you for your time.