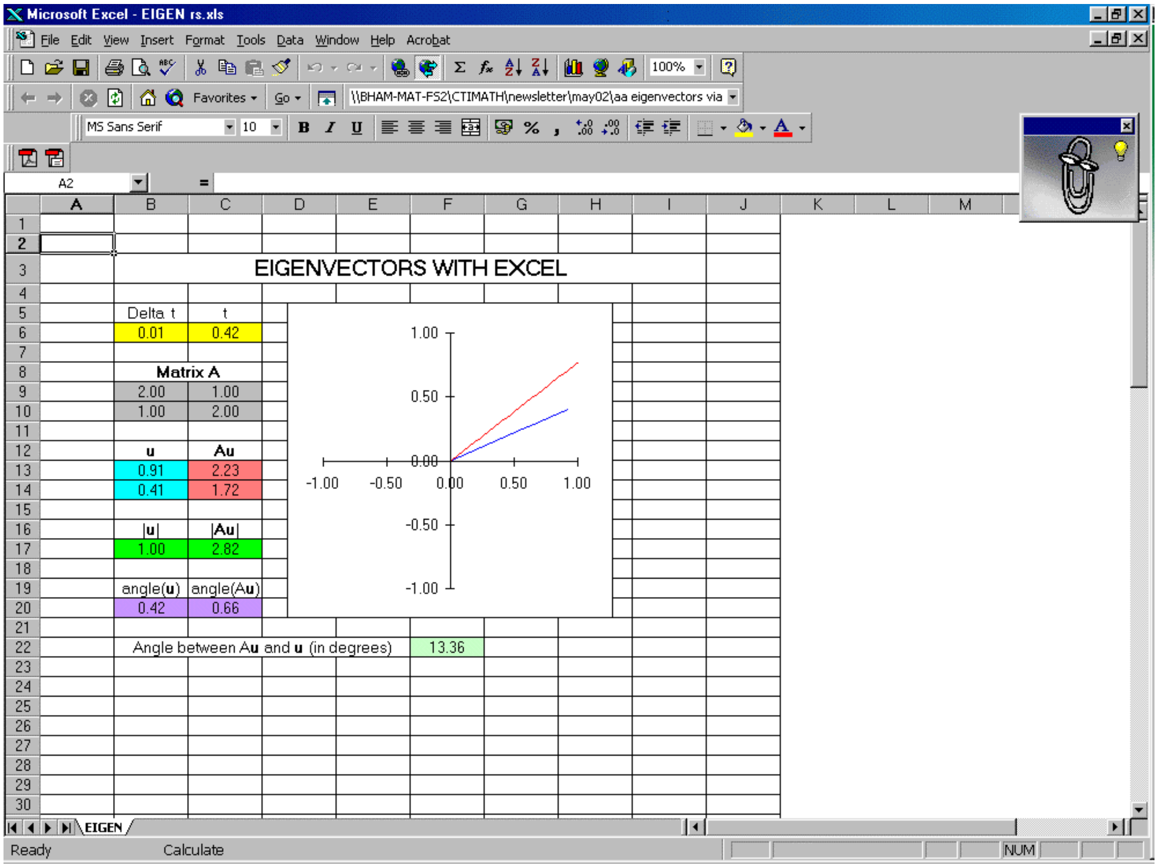
# AN INTRODUCTION TO EIGENVALUES AND EIGENVECTORS




Boot up EXCEL and load the file called EIGEN. You should find that your screen resembles the picture above. Let us go through the various elements contained in it.

The diagram shows a vector in blue. This is a unit vector radiating outwards from the origin. Let us call this vector **u**. The components of **u** are given in the dark blue cells (B13 and B14), with the magnitude of the vector **u** appearing below in cell B17. Below that (in cell B20) appears the angle (in radians) between this vector and the positive *x* axis, which we have called angle(**u**). Note that $-\pi <$ angle(**u**) $\leq \pi$ .

A matrix has been chosen. It is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. It has been called *A* and can be seen displayed in the grey cells (B9 – C10). Now when **u** is pre-multiplied by matrix *A*, it is transformed into a new vector *A***u**.
This vector is shown on the diagram in red. Its components are shown in the red cells (C13 and C14), with its magnitude shown below (C17) and the angle between this transformed vector and the positive *x* axis (again in radians) below that (C20). The angle

between the vector **u** and the transformed vector *A***u** is given (in degrees) in the green cell at the bottom (F22).

Now you will have noticed that the contents of one of the yellow cells (C6) are identical to those of one of the mauve cells (B20). This is no accident: *t* is the name we have given to the angle between **u** and the positive *x* axis. That is, angle(**u**) and *t* are the same thing.

You can change the value of *t*. Every time that you press the F9 key, the angle between **u** and the positive *x* axis will be increased by an amount 0.01 (delta *t*, the contents of the yellow cell B6 in the top left-hand corner). The vector **u** will move to its new position and EXCEL will calculate the new position of *A***u** and move the position on screen. The contents of the various cells will be updated. Try pressing F9 a few times now. If things are changing too fast (or too slowly), you can decrease (or increase) the value of delta *t*.

Now try, by repeated use of the F9 key, to arrive at a situation in which the red and blue vectors are pointing in the same direction. When this has been achieved, since **u** and *A***u** are parallel vectors, we have

$$A\mathbf{u} = \lambda\mathbf{u}, \quad \text{where } \lambda \text{ is some scalar.}$$

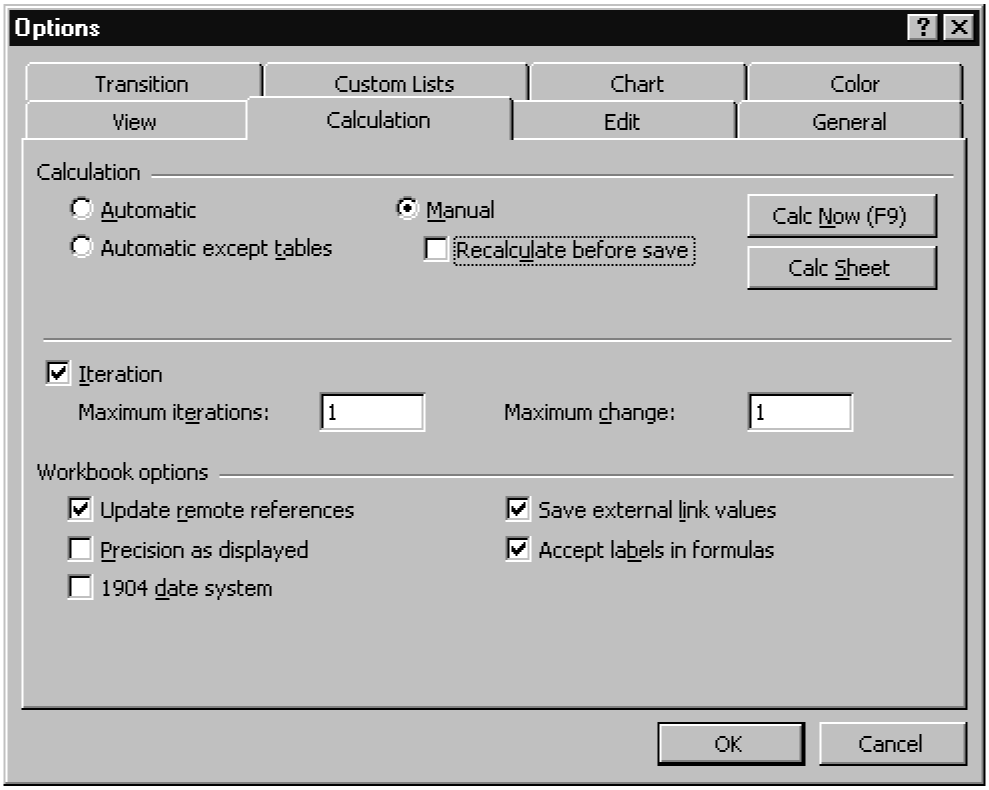
**u** is called an **eigenvector** of matrix *A* and $\lambda$ is the associated (or corresponding) **eigenvalue**. Make a note of the components of the eigenvector you have found and, find a way of calculating the eigenvalue $\lambda$ from the information which appears on screen.

Now use F9 again to see if you can arrive at a situation in which **u** and *A***u** are pointing in opposite directions. In such a case, **u** is an eigenvector with an associated eigenvalue which is negative. Make a note of the components of the eigenvector and, again, work out the corresponding eigenvalue from the information available on screen.

<u>Notes</u>

1. The diagram on the EXCEL screen is necessarily limited in size. The whole of the blue vector - the untransformed one - fits onto the diagram but the transformed vector - the red one - will not, in general, fit on the EXCEL diagram. This should not cause you any problems: you can see the magnitude of the transformed vector by taking a glance at the appropriate cell.

2. EXCEL must be set up so that the value of *t* will be incremented and the chart updated whenever the F9 key is pressed. If EXCEL is not already set up in this way, (for example, if pressing the F9 key produces no effect) then click on Tools Options Calculation and then make sure that EXCEL is set up as shown below.




Any scalar multiple of an eigenvector is also an eigenvector: it's the direction which is important, not the magnitude of the vector. If $A$ is any square matrix and $\mathbf{u}$ is an eigenvector with associated eigenvalue equal to $\lambda$, then we have

$$A\mathbf{u} = \lambda\mathbf{u}.$$

Now if $k$ is any scalar, you should be able to show that $k\mathbf{u}$ is an eigenvector (associated with the same eigenvalue). This is an easy calculation: begin with $A(k\mathbf{u})$ and show that it is equal to $\lambda(k\mathbf{u})$.

So, suppose that you find an eigenvector (of the given matrix) with the aid of the EXCEL chart and that it is given on the chart as $\begin{bmatrix} 0.71 \\ 0.71 \end{bmatrix}$, with $|A\mathbf{u}| = 2.99$, then it seems reasonable to assume that the eigenvalue is equal to 3 and to give the eigenvector in a simpler form, say $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The fact that this vector is indeed an eigenvector with corresponding eigenvalue equal to 3 is easily checked as follows.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

## Exercises

Find the eigenvectors and corresponding eigenvalues of each of the matrices below.

| | MATRIX | EIGENVALUE | EIGENVECTOR | EIGENVALUE | EIGENVECTOR |
|---|---|---|---|---|---|
| 1. | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ | | | | |
| 2. | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | | | | |
| 3. | $\begin{bmatrix} 1 & -1 \\ 1 & -3/2 \end{bmatrix}$ | | | | |
| 4. | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | | | | |
| 5. | $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ | | | | |
| 6. | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ | | | | |
| 7. | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | | | | |
| 8. | $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ | | | | |
| 9. | $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ | | | | |
| 10. | $\begin{bmatrix} 1 & -2 \\ 8 & 11 \end{bmatrix}$ | | | | |