Interactive Excel Investigations in Management Mathematics

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As a part of students' projects two MS Excel packages "MaxAlgebra" and "AP" have been created in the School of Mathematics and Statistics at the University of Birmingham in support of interactive experiments related to some unresolved combinatorial problems.

1. MaxAlgebra (Student A)

Max-algebra, where the arithmetic operations of addition and multiplication are replaced by

$$a \oplus b := \max(a, b) \text{ and } a \otimes b := a + b$$

offers an attractive way for modelling some types of discrete-event dynamic systems and optimisation problems in production and transportation.

The following simple example indicates the nature of practical problems in management mathematics for which max-algebra is a suitable formulation language:

Suppose two trains leave two different stations but arrive at the same station from which a third train, connecting to the first two, departs. Let us denote the departure times of the trains as x_1, x_2 and x_3 respectively, and the duration of the journeys of the first two trains (including the necessary times for changing the trains) by a_1 and a_2 respectively. Then

$$x_3 = \max(x_1 + a_1, x_2 + a_2)$$

which in the max-algebraic notation reads

$$x_3 = x_1 \otimes a_1 \oplus x_2 \otimes a_2$$

thus it is a max-algebraic scalar product of the vectors (x_1, x_2) and (a_1, a_2). So if the departure time of the third train is given say, as constant b, and we need to find the necessary departure times of the first two trains, we have to solve a simple max-algebraic linear equation

$$x_1 \otimes a_1 \oplus x_2 \otimes a_2 = b\,.$$

We can extend the pair of operations $(\oplus, \otimes)$ to matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices or vectors over $\boldsymbol{R}$ of compatible sizes then we write

$$C = A \oplus B \text{ if } c_{ij} = a_{ij} \oplus b_{ij} \text{ for all } i, j$$

and $$C = A \otimes B \text{ if } c_{ij} = \sum\nolimits_k^{\otimes} a_{ik} \otimes b_{kj} \text{ for all } i, j$$

This enables us to deal with the systems of linear equations

$$A \otimes x = b$$

which can be used for instance to model the task of starting times in a machine-scheduling problem, the eigenproblem

$$A \otimes x = \lambda \otimes x$$

which corresponds to a synchronisation problem in industrial production and the permanent of a matrix

$$per(A) = \max_{\pi \in P_n} \sum_{i \in N} a_{i,\pi(i)}$$

which corresponds to the linear assignment problem (see AP below).

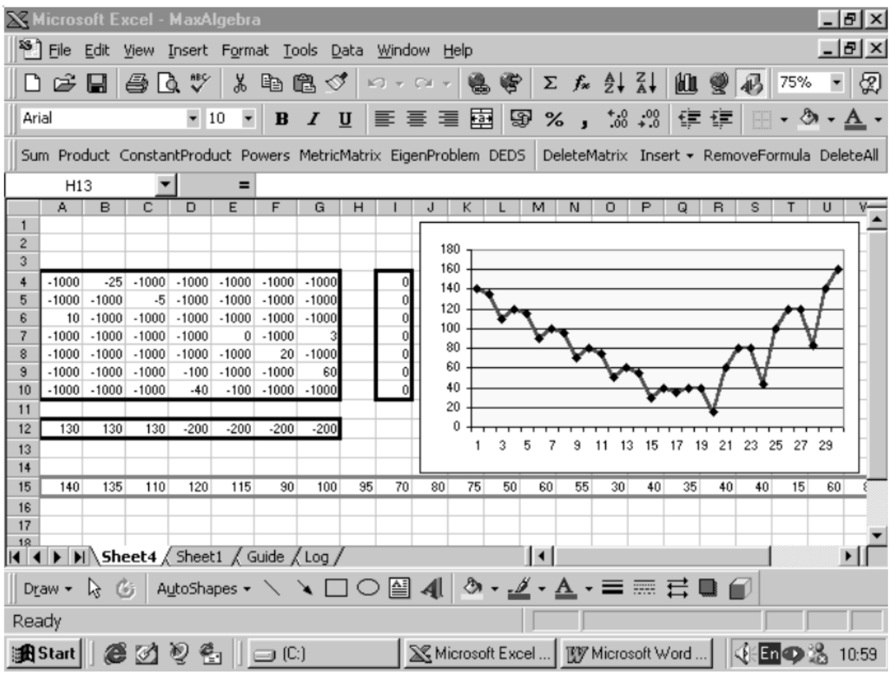
Intensive research [1,3] in this area began in the 1960's and was devoted to a wide range of problems such as linear systems, eigenvalue/eigenvector problem, linear independence, regularity, rank and dimension, characteristic polynomial and more recently discrete-event dynamic systems and combinatorial aspects of max-algebra. Some of these problems (such as linear systems, permanent, factorisation of polynomials) are easier than their linear-algebraic counterparts, some of them are more difficult (for instance regularity, characteristic polynomial). Some of the difficult problems are essentially related to the question of realisibility of a discrete-event dynamic system [1,2,4].

Realisibility is the following classical problem: An unknown system emits a sequence of real-number signals $G=\{g_j\}_0^\infty$ at discrete time intervals. Find a compact description of the system given only this observed sequence. Solutions of this problem depend on the assumption we set on the underlying process. In system theory significant effort has been devoted to the case when the process is describable through the state vector $x(j) \in \boldsymbol{R}^n$ of the system at time $j = 0, 1, \ldots$ and the change of state is described through a linear transformation. In some synchronous processes this would become a max-algebraic linear transformation of the form

$$x(j) \rightarrow x(j+1) = A \otimes x(j), \quad x(0) = b,$$

where $A \in \boldsymbol{R}^{n \times n}$, $b \in \boldsymbol{R}^n$ and the states are observed through an observation vector $c \in \boldsymbol{R}^n$ that is

$$g_j = c^T \otimes x(j), \quad j = 0, 1, \ldots$$



We call g_j Markov parameters and any triple (A, b, c), where $A \in \boldsymbol{R}^{n \times n}$, $b, c \in \boldsymbol{R}^n$, a realisation of the discrete-event dynamic system (DEDS) emitting G if

$$g_j = c^T \otimes A^{(j)} \otimes b, \ (j = 0, 1, \ldots)$$

In general, there are many trivial realisations but for a compact, economical description we seek a realisation of the least possible dimension n – called a minimal dimensional realisation (MDR). There are some particular results in this area, especially for some special types of sequences but a more general solution idea is still missing.

One way of tackling this type of problems is to experiment with sequences, for instance to generate DEDS using various triples (A, b, c), to observe the behaviour of Markov parameters depending on the changes of entries in (A, b, c) and then to form conjectures which a researcher could try to prove or disprove.

The package MaxAlgebra is following this idea: to create a workplace in which it would be possible in a convenient and efficient way to experiment with DEDS. This means to enter the data of the triple (A, b, c) in an efficient manner, generate the sequence of Markov parameters, numerically and graphically, and then observe the effect of the changes of the input parameters on the resulting sequence without the necessity to re-enter the whole input. The package also contains subroutines for basic max-algebraic operations, including the product of two matrices, powers of a square matrix, finding the metric matrix and the (max-algebraic) eigenproblem solver, as these may be needed during experiments with the DEDS. Among others the program has the options of automatic generation of Hankel matrices, generation of random matrices and keeping or removing links between the input data and calculated results.

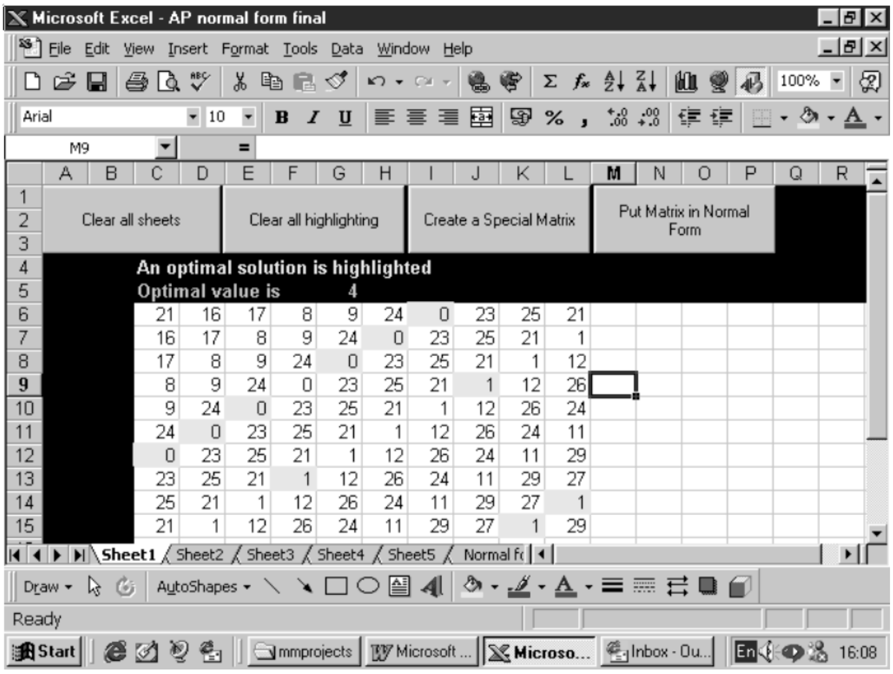
2. AP (Student B)

This program has been made in support of investigations of the solution set of the linear assignment problem which can be formulated as follows:

Given an $n \times n$ matrix $A = (a_{ij})$, find n elements in this matrix, one in each row and column, so that their sum is minimal [maximal].

Formally, we have to find a permutation π of the set $\{1, \ldots, n\}$ minimising [maximising] the sum

$$\sum_{i=1}^{n} a_{i,\pi(i)}$$



This problem has numerous applications in operational research. It is known that although the assignment problem itself is efficiently solvable (probably the best known solution method is the Hungarian method), there are some variants of this problem, important for solving the realisibility problem, that remain hard [2]. These questions are related to the existence of certain types of optimal permutations. A so called AP-normal form of a matrix *A* is a matrix *B* which can be obtained from *A* by adding suitable constants to the rows and/or columns, is non-negative [non-positive] and it contains *n* zeros no two of which are in the same row or column. This form is an efficient and transparent way of describing sets of optimal solutions as it uses n^2 elements to describe the set of size up to $n!$. So the aim of this package was to provide a user-friendly environment using which a researcher can investigate the properties of the solution set to the assignment problem for a matrix using its normal form.

The package does not actually deal so much with the solving of the assignment problem (although this is also done as a "by-product") as with the finding of AP-normal forms. It uses the max-algebraic eigenvectors for scaling the matrix to its AP-normal form and hence it produces n normal forms (not necessarily distinct). A feature of the program very important in connection with DEDS is that it enables fast entry of Hankel matrices, including generation of random Hankel matrices.

Both packages are written in Visual Basic and special toolbars have been created for the ease of communication (see the figures). The main buttons on the MaxAlgebra toolbar are "DEDS" and "Eigenproblem", the others are designed to provide support for the main calculations. Detailed specification of required operations is entered using dialogue boxes.

Both programs show good performance for matrices of order up to 30. MaxAlgebra is accompanied by a manual. Packages are available for any interested researcher and can be obtained from the author, who was the supervisor of these projects.

References:

[1] F L Baccelli, G Cohen, G-J Olsder and J-P Quadrat (1992), *Synchronization and Linearity*, JWiley and Sons

[2] P Butkovic (1995) *Regularity of matrices in min-algebra and its time-complexity*, Discrete Appl Math 57, 121-132

[3] R A Cuninghame-Green (1979) *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems 166, Springer, Berlin

[4] S Gaubert, P Butkovic and R A Cuninghame-Green (1998). *Minimal (max, +) realization of convex sequences*, SIAM J Control and Optimization 36, No 1, 137-147